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Device of Non-Classical Newton's Minorant of Functions of Two Real Table-like Variables and its Application in Numerical Analysis

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ABSTRACT

A device of non-classic Newton's minorant and their graphs of functions of two real table-like variables have been introduced and a new numerical method for finding extremum both smooth and non-smooth logarithmically convex functions of two real variable has been constructed.

Keyword: *minorant of function, numerical analysis, optimization method.*

I. INTRODUCTION

In [1] a device of non-classical Newton's majorants and diagrams of functions of one and two real variables given in tabular form is constructed and its usage for the development of numerical methods for solving the Cauchy problem for ordinary differential equations and their systems, accurate to a certain class of functions is discussed. In [2] this device is used to construct numerical methods for non-smooth optimization logarithmically concave functions of one, two or several real variables. In [3] the device of non-classical Newton's minorant and their graphs of functions of one real variable given in tabular form is constructed, which in [4] is used to develop numerical methods for non-smooth optimization logarithmically convex functions of one and two real variables.

This paper describes the construction of the device of non-classical Newton's minorant and their graphs of functions of two real variables defined in tabular form, and its usage for the development of numerical optimization method non-smooth functions of two real variables.

II. DEVICE OF NON-CLASSICAL NEWTON'S MINORNT OF FUNCTIONS OF TWO REAL TABLE-LIKE VARIABLES

Let us consider the function of two real variables $z = f(x,y)$, which is given values at the points (x_i, y_j) $(i = 0,1,\dots,n; j = 0,1,\dots,m)$:

$$f(x_i, y_j) = z_{ij} \quad (i = 0,1,\dots,n;\ j = 0,1,\dots,m) \qquad (1)$$

Let $x_0 < x_1 < \cdots < x_n,\ y_0 < y_1 < \cdots < y_m$ i

$$|z_{ij}| = a_{ij} \le M \quad (i = 0,1,\dots,n; j = 0,1,\dots,m), \qquad (2)$$

where M – certain constant.

Point $P_{ij}(x_i, y_j, -\ln a_{ij})$ coordinates $x = x_i,\ y = y_j,\ z = -\ln a_{ij}$ in space xyz will be called bitmaps value function $f(x,y)$ in the point (x_i, y_j).

Assume that the points of the image P_{ij} of the function $z = f(x,y)$ at points (x_i, y_j) $(i = 0,1,\dots,n;\ j = 0,1,\dots,m)$ in space xyz are built. From every point P_{ij} we will draw a semi-direct in negative direction of the axis Oz perpendicular to the plane xy. The set of points of the semi-directs is denoted as S, and its convex hull – through $C(S)$. For each point $(x,y)\epsilon R$,

where

$$R = \{x_0 \le x \le x_n,\ y_0 \le y \le y_m\}$$

let us define the point $B(x, y, \chi(x,y))$, where

$$\chi(x,y) = \sup_{(x,y,z)\epsilon C(S)} z.$$

The set of points $B(x, y, \chi(x,y))$, which $(x,y)\epsilon R$, forms a multi-faceted surface δ_f, that limits $C(S)$ the top. This surface is continuous, concavity and its equation is

$$z = \chi(x,y),\ (x,y)\epsilon R.$$

The surface δ_f has the following properties:

- each vertex δ_f is placed in one of the pixel P_{ij} values of the function $z = f(x,y)$ in the point (x_i, y_j);
- each imagine point P_{ij} is located on δ_f or below it;
- each point (x_i, y_j) corresponds to the point $B_{ij}(x_i, y_j, \chi_{ij})$ surface δ_f, where

$$\chi_{ij} = \chi(x_i, y_j).$$

Let

$$m_f(x,y) = \exp(-\chi(x,y)), \qquad (x,y)\epsilon R.$$

Then for each point (x_i, y_j) $(i = 0,1,\dots,n; j = 0,1,\dots,m)$ the inequality is performed

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$$m_f(x_i,y_j) \le |f(x_i,y_j)| = |z_{ij}| = a_{ij}.$$

Indeed, with the construction it δ_f follows that

$$-\ln a_{ij} \le \chi(x_i,y_j),$$

or

$$a_{ij} \ge \exp\left(-\chi(x_i,y_j)\right) = m_f(x_i,y_j).$$

In addition,

$$m_f(x_0,y_0) = |f(x_0,y_0)|, m_f(x_0,y_m) = |f(x_0,y_m)|,$$
$$m_f(x_n,y_0) = |f(x_n,y_0)|, m_f(x_n,y_m) = |f(x_n,y_m)|.$$

Function $m_f(x,y)$, defined on R, is called non-classical Newton's minorant of function $z = f(x,y)$, and $\delta_f -$ its surface chart.

Let

$$m_f(x_i,y_j) = t_{ij},\ (i = 0,1,\dots,n;\quad j = 0,1,\dots,m).$$

Values

$$r_{ij}(x) = \left(\frac{t_{i-1,j}}{t_{ij}}\right)^{\frac{1}{x_i - x_{i-1}}}$$
$$(i = 1,2,\dots,n;\quad j = 0,1,\dots,m)$$

and

$$r_{ij}(y) = \left(\frac{t_{i,j-1}}{t_{ij}}\right)^{\frac{1}{y_j - y_{j-1}}}$$
$$(j = 1,2,\dots,m;\quad i = 0,1,\dots,n)$$

are called (i,j)-th numerical inclinations of Newton's minorant $m_f(x,y)$ respectively, in the direction of the axes of abscissas and ordinates, and values

$$d_{ij}(x) = \frac{r_{i+1,j}(x)}{r_{ij}(x)}$$
$$(i = 1,2,\dots,n-1;\quad j = 0,1,\dots,m;\quad d_{0j} = d_{nj} = \infty)$$

and

$$d_{ij}(y) = \frac{r_{i,j+1}(y)}{r_{ij}(y)}$$
$$(j = 1,2,\dots,m-1;\quad i = 0,1,\dots,n;\quad d_{i0} = d_{im} = \infty)$$

are called (i,j)-th disabilities of Newton's minorant $m_f(x,y)$ respectively, in the direction of the axes Ox and Oy.

From concavity Newton's diagram δ_f the following inequalities are obtained:

$$r_{ij}(x) \ge r_{i+1,j}(x);\quad (i = 0,1,\dots,n-1;\quad j = 0,1,\dots,m);$$
$$r_{ij}(y) \ge r_{i,j+1}(y);\quad (i = 0,1,\dots,n;\quad j = 0,1,\dots,m-1);$$
$$d_{ij}(x) \le 1;\quad (i = 1,2,\dots,n-1;\quad j = 0,1,\dots,m);$$
$$d_{ij}(y) \le 1;\quad (i = 0,1,\dots,n;\quad j = 1,2,\dots,m-1).$$

If the point P_{ij} is located at the top of the δ_f, then a couple of indexes (i,j) are called vertex, and if it is placed on δ_f, then it is called a couple of chart symbols.

III. NUMERICAL METHOD OF MINORANT TYPE FOR FINDING EXTREMUM RANDOM LOGARITHMICALLY CONVEX FUNCTION OF TWO REAL VARIABLES

Let in $D = \{a \le x \le b, c \le y \le d\}$ a logarithmically convex function $f(x,y)$ is defined, that can be either smooth or rough. Let us construct in D grid:

$$x = x_i = a + ih;\quad i = 0,1,\dots,n;\quad h = \frac{b-a}{n};$$
$$y = y_j = c + js;\quad j = 0,1,\dots,m;\quad s = \frac{d-c}{m}.$$

Let denote

$$f(x_i,y_j) = a_{ij}\ \ (i = 0,1,\dots,n; j = 0,1,\dots,m).$$

Construct for the values $f(x_i,y_j) = a_{ij}$ non-classical Newton's minorant $m_f(x,y)$. Since the function $f(x,y)$ is a logarithmically convex in D, then

$$m_f(x_i,y_j) = a_{ij}\quad (i = 0,1,\dots,n; j = 0,1,\dots,m).$$

Then numerical inclinations and disabilities of Newton's nimorant $m_f(x,y)$ are determined by the formulae:

$$r_{ij}(x) = \left(\frac{a_{i-1,j}}{a_{ij}}\right)^{\frac{1}{x_i - x_{i-1}}}$$
$$(i = 1,2,\dots,n;\quad j = 0,1,\dots,m)$$
$$r_{ij}(y) = \left(\frac{a_{i,j-1}}{a_{ij}}\right)^{\frac{1}{y_j - y_{j-1}}}$$
$$(j = 1,2,\dots,m;\quad i = 0,1,\dots,n)$$

$$d_{ij}(x) = \frac{r_{i+1,j}(x)}{r_{ij}(x)}$$
$$(i = 1,2,\dots,n-1;\quad j = 0,1,\dots,m;\quad d_{0j} = d_{nj} = \infty)$$
$$d_{ij}(y) = \frac{r_{i,j+1}(y)}{r_{ij}(y)}$$
$$(j = 1,2,\dots,m-1;\quad i = 0,1,\dots,n;\quad d_{i0} = d_{im} = \infty)$$

Algorithm of the method. We should firstly note that if for some point $(x_k,y_l)\epsilon D$ the conditions are accomplished

$$r_{kl}(x) \ge 1,\qquad r_{k+1,l}(x) < 1;\qquad (3)$$
$$r_{kl}(y) \ge 1,\qquad r_{k,l+1}(y) < 1,\qquad (4)$$

then the point (x_k,y_l) with accuracy $\varepsilon = \max(s,h)$ is a point in the function extremum $f(x,y)$.

If the condition (1) is not satisfied, then

$$r_{kl}(x) > 1,\qquad r_{k+1,l}(x) \ge 1;\qquad (5)$$

or

$$r_{kl}(x) \le 1,\qquad r_{k+1,l}(x) < 1;\qquad (6).$$

In the default condition (2)

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$$r_{kl}(y) > 1, \qquad r_{k,l+1}(y) \geq 1; \qquad (7)$$

or

$$r_{kl}(y) \leq 1, \qquad r_{k,l+1}(y) < 1; \qquad (8).$$

By selecting an initial approximation of an extreme point any point $(x_k, y_l)\epsilon D$, algorithm of the method is as follows:

1. If for the point (x_k, y_l) conditions (3),(4) are performed, then the point (x_k, y_l) with accuracy $\varepsilon \leq \max(s, h)$ is accepted as an optimal and the algorithm is terminated.
2. If for the point (x_k, y_l) condition (3) is not performed and condition (4) is performed, then in case (5) we find the smallest value of index $\mu \geq 1$, for witch $r_{k+\mu,l}(x) < 1$; in case (6) we find the smallest value of index $\nu \geq 1$, for which $r_{k-\nu,l}(x) \geq 1$. Having found the closest point to the point (x_k, y_l) for which condition (3) is performed and marked this point by (x_k, y_l), we turn to step 1
3. If for the point (x_k, y_l) condition (3) is performed and condition (4) is not performed, then is case (7) we find the smallest value of index $\mu \geq 1$, for which $r_{k,l+\mu}(y) < 1$; in case (8) we find the smallest value of index $\nu \geq 1$, for which $r_{k,l-\nu}(y) \geq 1$. Having found the closest point to the point (x_k, y_l) for which condition (4) is performed and marked this point by (x_k, y_l), we turn to step 1.
4. If for point (x_k, y_l) conditions (3), (4) are not performed, then similar as in step 2 we find closest point to the point (x_k, y_l) for which condition (3) is performed. Denoting this point by (x_k, y_l), similarly to step 3 we find the closest point to the point (x_k, y_l) for which condition (3) is performed. Denoting this point by (x_k, y_l), we turn to step 1.

If greater accuracy is necessary to find an extreme point, we take a neighbourhood of a point found for region *D*, reduce steps *h* and *s* and perform the algorithm.

We will demonstrate a comparison of this method with Newton's method for example penalty function #2 with *n=2:*

$$f(x, y) = (x-1)^2 + (y-1)^2 + 10^{-3}(x^2 + y^2 - 0.25)^2$$

Graph of this function is shown in fig.1.

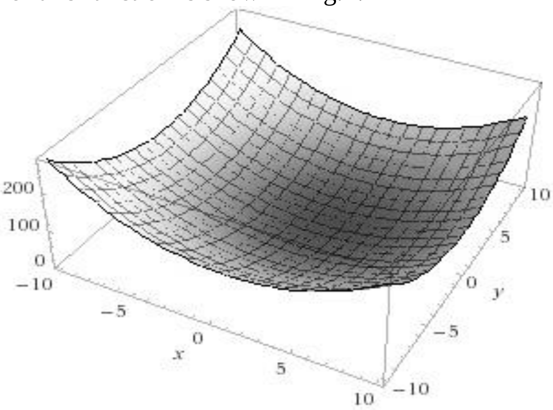

Fig.1: The graph of the penalty function №2

Let *h*=*s*=0.01. Choose a point (4; 3) for the start point. The sequence of iterations is given in table 1:

Table 1

№	x	y	$f(x)$
0	4	3	13,612563
1	0,98	1	0,0033255
2	1	1	0,0030625

So with accuracy 0,01 point (1; 1) we accept the point at which the function reaches its minimum:

$$f(x, y) = 0{,}0030625.$$

Choose again *h*=*s*=0.01 and the starting point also choose the point (4;3). The sequence of iterations for Newton's method is given in table 2:

Table 2

№	x	y	$f(x)$
0	4	3	13,612563
1	1,205	1,214	0.094981
2	0,997	0,997	0.0030387

So with accuracy 0,01 point (0,997; 0,997) we accept the point at which the function reaches its minimum:

$$f(x, y) = 0.0030387.$$

IV. CONCLUSIONS

The device of non-classical Newton's minorant of functions of two real variables defined in tabular form, which is used to develop new numerical method for finding extremum as smooth and non-smooth logarithmically convex functions of two real variables, has been constructed. The main advantage of this method is that the convergence of the method is independent of the choice of the initial approximation. There is no need to know the neighborhood of point of extremum and optimization function can be either smooth or rough, discontinuous and discrete. Considering also that the usage of methods that use derivatives, such as Newton's method, encounters obstacles: it is necessary to know the derivative of the function that minimize (may be a situation where it becomes difficult to get the derivative as an analytic function), then the benefits of this method can be attributed to its simplicity.

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